

Problem 2.24

[Difficulty: 3]

2.24 A velocity field is given by $\vec{V} = ax\hat{i} - by\hat{j}$, where $a = 0.1 \text{ s}^{-2}$ and $b = 1 \text{ s}^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0 \text{ s}$, plot the pathline during the interval from $t = 0$ to $t = 3 \text{ s}$. Compare with the streamlines plotted through the same point at the instants $t = 0, 1$, and 2 s .

Given: Velocity field

Find: Plot pathlines and streamlines

Solution:

Pathlines are given by $\frac{dx}{dt} = u = a \cdot x \cdot t$ $\frac{dy}{dt} = v = -b \cdot y$

So, separating variables $\frac{dx}{x} = a \cdot t \cdot dt$ $\frac{dy}{y} = -b \cdot dt$

Integrating $\ln(x) = \frac{1}{2} \cdot a \cdot t^2 + c_1$ $\ln(y) = -b \cdot t + c_2$

For initial position (x_0, y_0) $x = x_0 \cdot e^{\frac{a}{2} \cdot t^2}$ $y = y_0 \cdot e^{-b \cdot t}$

Using the given data, and IC $(x_0, y_0) = (1, 1)$ at $t = 0$

$x = e^{0.05 \cdot t^2}$ $y = e^{-t}$

Streamlines are given by $\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot y}{a \cdot x \cdot t}$

So, separating variables $\frac{dy}{y} = -\frac{b}{a \cdot t} \cdot \frac{dx}{x}$ Integrating $\ln(y) = -\frac{b}{a \cdot t} \cdot \ln(x) + C$

The solution is $y = C \cdot x^{-\frac{b}{a \cdot t}}$

For streamline at $(1, 1)$ at $t = 0 \text{ s}$ $x = c$

For streamline at $(1, 1)$ at $t = 1 \text{ s}$ $y = x^{-10}$

For streamline at $(1, 1)$ at $t = 2 \text{ s}$ $y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

t = 0	
x	y
1.00	1.00
1.00	0.78
1.00	0.61
1.00	0.47
1.00	0.37
1.00	0.29
1.00	0.22
1.00	0.17
1.00	0.14
1.00	0.11
1.00	0.08
1.00	0.06
1.00	0.05
1.00	0.04
1.00	0.03
1.00	0.02
1.00	0.02
1.00	0.01
1.00	0.01
1.00	0.01
1.00	0.01

t = 1 s	
x	y
1.00	1.00
1.00	0.97
1.01	0.88
1.03	0.75
1.05	0.61
1.08	0.46
1.12	0.32
1.17	0.22
1.22	0.14
1.29	0.08
1.37	0.04
1.46	0.02
1.57	0.01
1.70	0.01
1.85	0.00
2.02	0.00
2.23	0.00
2.47	0.00
2.75	0.00
3.09	0.00
3.49	0.00

t = 2 s	
x	y
1.00	1.00
1.00	0.98
1.01	0.94
1.03	0.87
1.05	0.78
1.08	0.68
1.12	0.57
1.17	0.47
1.22	0.37
1.29	0.28
1.37	0.21
1.46	0.15
1.57	0.11
1.70	0.07
1.85	0.05
2.02	0.03
2.23	0.02
2.47	0.01
2.75	0.01
3.09	0.00
3.49	0.00

